A new Reading to Scaffolding in Geometry

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Abstract:
This paper aims at suggesting a solution to the difficulties encountered in Geometry teaching – learning and specifically in applying the acquired geometric skills to solve real life problems. Based on Bruner's Scaffolding, Van Hiele's model of development of Geometric growth, and Kuzniak's three levels of Geometry learning, intermediate school-Grade 9 students should be exposed to real life problems that can be solved using Geometry. In this approach, students are scaffold with suitable hints and clues in adequate times in order to progress and solve a sample problem with the aid of the teacher to pass through the Zone of Proximal Development (ZPD) level. As the student crosses her/his ZPD region, s/he is evaluated through a more advanced real life problem in order to check for the validity of the hypothesis which turned out to be successful with appreciable percentage. Therefore, the intermediate school books are requested to be revised to add the necessary well – planned real life problems with at least a sample problem and another for evaluation; and the in-service Math teachers are requested to solve such problems in class to fulfill the requirements of the deemed changes in curriculum.

Keywords: Geometry in Middle School, Scaffolding, Real life Geometry Problems

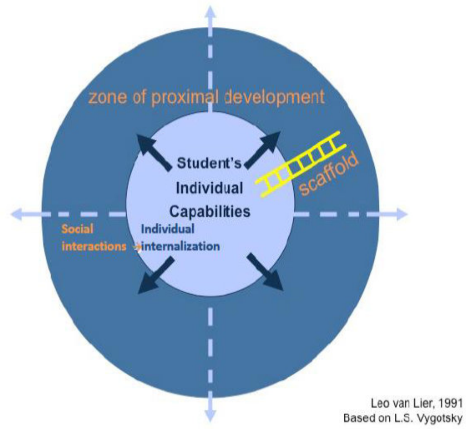
Introduction

Geometry learning skills are deemed necessary in everyday life; learners in grade 9 level are faced with all the difficulties of learning the Geometric skills they need in their future studies and in their future lives to solve daily situation problems like guiding a helicopter to put off a fire with the mean time possible or helping Nadim (Grade 9 virtual student) to move from one side of a river onto a bridge to the other side of the river following the shortest path possible. Solving such problems requires mastering of geometric theorems and skills already 'supposedly' acquired by Grade 9 students. Unfortunately, most of the students fail to accomplish the job in helping the pilot or Nadim to choose the optimum solution.

The objective of this paper is to highlight on different approach integrating Jerome Bruner's thoughts and theories of Scaffolding and Spiral Curriculum with Pierre Van Hiele's sequential model of learning

Geometry in addition to Alan Kuzniak's overview of the three levels in Geometry learning to plan instruction of Geometry classes for Grade 9. The three aforementioned poles, Bruner – Van Hiele – and Kuzniak, share common ideas about intuitive learning, boosting learners' motivations, and providing guidance (in correspondence with Zone of Proximal Development – ZPD – as per Lev Vygotsky).

The adjacent figure explains Lev Vygotsky's definition of the Zone of proximal development. According to Vygotsky, The area or region where the learner achieves with the help of the more knowledgeable other what he couldn't do alone before and will be able to solve after the passage in the ZPD margin. This is also scaffolding to move from student's individual capabilities to student's future developed capabilities. (Walqui and Strom, n.d. P2)



Within the ZPD, four types of scaffolding interactions are possible:

Assistance from an expert other	Collaboration with an equal peer	Collaboration with a less capable peer	Use of internal resources
Learner receives guidance, advice, and/or modeling from teacher or more capable peer.	Learner works together with a student of the same capabilities to construct meaning together.	Learner assists lower-level learner, which provides the opportunity to articulate, refine, and expand their own knowledge.	Learner works independently, relying on their own internalized practices, strategies, inner speech, and other resources.

(Walqui and Strom, n.d. P3)

The above table displays different types of Scaffolding that can take place according to student's needs in order to accomplish development.

Hereby, we formulate our hypothesis as: guidance in solving geometric problems does not only include mastery of geometric rules in direct application but also holds the drilling practice aided by well-designed hints/clues in order to provide the necessary demonstration and the relevant examples to be stored in learners' long term memory and retrieved whenever needed in the future.

Literature Review:

Many authors, teachers, publishers… were concerned with the difficulties students encounter during elementary, middle and high school development of Geometry skills. In what follows a brief review is presented on the ideas various articles approached concerning the topic of this paper.

1- An article, written by J.E. Schwartz – Pearson Allyn Bacon Prentice Hall, (last update was in July 2010) under the title: "Why Do People Have Difficulty with Geometry?", argues that most people carry with them their bitter memories of the hard days of Geometry Learning without comprehension and that the low level experiences in the inadequate geometry curriculum is accused to hold the full responsibility.

The author disagrees with J. Piaget stages of cognitive development and is fond of Van Hiele sequenced levels to develop the necessary geometric skills or thinking. He emphasizes on the crucial role of appropriate experiences with geometric concepts and not on maturation. Schwartz suggests designing geometric activities to which the learner should be exposed during the middle and high school levels in order to compensate for the lack of such experiences in the present curricula and the admitted books.

2- Another article by Hanan Al Masri – LAU Lebanese American University – (2013), titled: “Difficulties of construction and formulating geometric proofs by Lebanese middle school students learning math in English.” discusses the topic from a different perspective. She conducted a case study in one of the Lebanese Schools in Beirut over a duration corresponding to one academic year in order to determine the factors behind the difficulties of geometric proofs. The results of her study showed that there are many factors affecting the problem; out of which I mention: “Understanding the Notion of Proof (UNP), Setting Proof Plans (SPP), Conducting Deductive Reasoning (CDR), and Understanding-Applying Mathematical Concepts (UAMC).”

3-Geoffrey Towell and Richard Lehrer, (1992) article: “A knowledge-Based Model of Geometry Learning” discusses that learners- in primary level classes- should be trained with similar materials to acquire the necessary Geometric skills and progress to develop geometry thinkers.

4- ICME13, a conference to be held in Hamburg in 2016 under the theme: “Teaching and Learning Geometry (Primary Level)” shows the importance of the topic of this paper as it invites researchers to write their papers about the following topics in addition to other topics:

- “Application of Geometry in the real world and other subjects;
- Explanation, Argumentation, and Proof in Geometry education;
- Teacher preparation in Geometry Education.” (ICME13.org)

5**-** “Performance on Middle School Geometry Problems with Geometry Clues Matched to Three Different Cognitive Styles” an article written by: Karen L. Anderson, M. Beth Casey, William L. Thompson, Marie S. Burrage, Elizabeth Pezaris, and Stephen M. Kosslyn - Harvard University's DASH Repository – (2008) reflects on students’ performance in geometry problems when provided with different clues and based on three ability-based cognitive styles (verbal deductive, spatial and object imagery).

An assessment was done to classify the students’ cognitive styles and the research came out with emphasis that spatial imagery and verbal deductive cognitive styles were necessary to solve geometry problems.

6- The New York State Common Core, (2014) published Mathematics Curriculum for grade 3 – module 7 for Geometry and Measurement Word Problems. Under this topic, Scaffolds were highlighted as part of their strategies in solving problems. Scaffolds are presented as specific margin notes at appropriate times for the student to think of and implement to solve a problem.

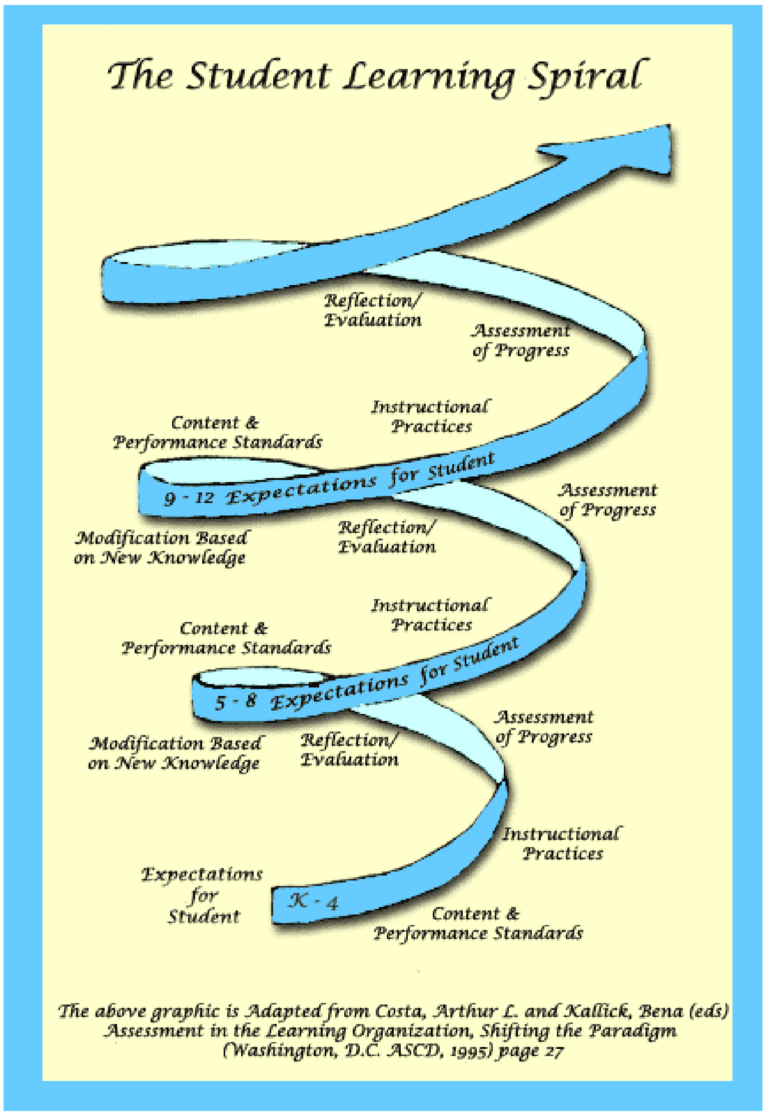
The presented articles emphasize the importance of the problems faced while learning Geometry especially in the middle school classes. The problem can be considered international as each of the articles was written in a different region around the world; and consequently alleviates the importance of the subject. The articles also reflect that researchers are continuously looking for better resolutions to the problem. Finally, researchers are going back to the roots of the issue and they anchor on stable foundations as the scaffolds and ideas of Bruner, Van Hiele, Kuzniak and others to build their suggested solutions.

The Theoretical part:

As clearly indicated in the introduction, this part presents the theories of J. Bruner, P. Van Hiele, and A. Kuzniak and how their findings can be related to the topic.

Jerome Bruner, the famous psychologist, whose works and books contributed to learning-education processes. In his Book “The Process of Education” (1960), Bruner discussed where learning starts for each individual. Learning starts at the level of cognition of each learner – in order to apply constructivism and socio-constructivism basics of learning: the learner constructs new knowledge on previously acquired ones. (Research for Teachers, 2006).

Bruner also came across the transfer through appropriate learning that will ultimately lead to ‘learning how to learn’.



http://www.nyu.edu/classes/gilbert/musedtechwkshp/learn.html

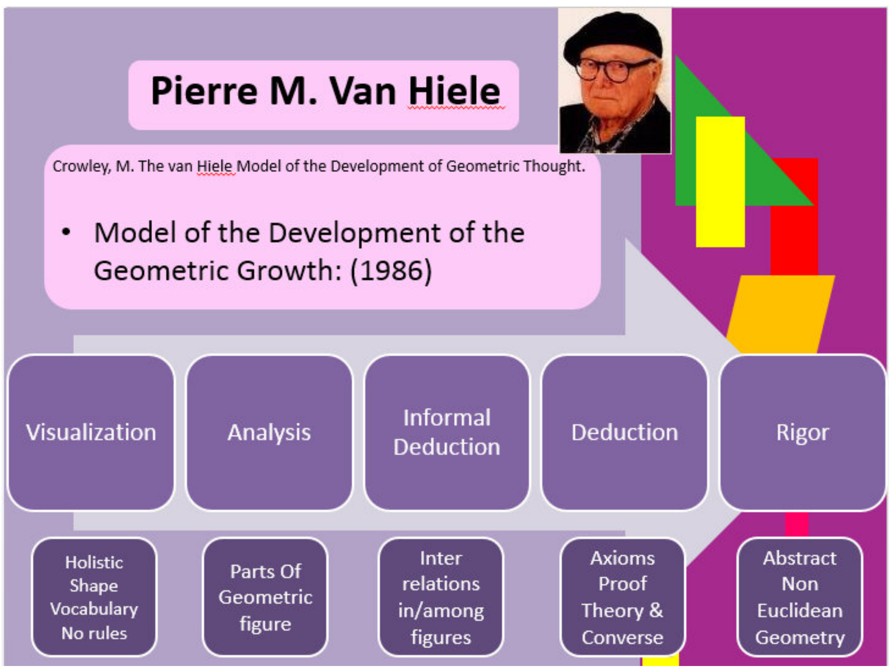
Bruner strongly advised for guided inquiries to accelerate students’ thinking. In this respect, Bruner focused on the spiral curriculum which can be explained as follows: learners acquire the basic ideas initially by using their intuition; and after words, the learner builds on them by revisiting these basic ideas as frequent as required until the meaningful understanding is fully achieved. He provided different examples on group work projects done in class and guided by the teacher/facilitator in different disciplines as language, history, and geography. The projects were designed to motivate the students towards learning as such projects are planned on real life examples which by nature drags the students' attention and enrolls him in finding an optimal solution.

Bruner tackled four important pillars: motivating the student, knowing how much the student is ready to learn, structuring the learning process, and encouraging student's intuitive and analytical thinking.

Under structuring the learning process, Bruner supported Vygotsky's perspective of the Zone of Proximal Development (ZPD) where the learner is guided and provided with scaffolds in order to cross the region where learner needs the help of the more knowledgeable other and consequently becomes able to solve similar problems on his own. Therefore, a crucial step labeled as Scaffolds is crucial in the learning process. (Bruner, 1976, Kolb, 1984, and Vygotsky, 1978)

Pierre Van Hiele was able between 1957 and 1986 to create and elaborate a model for the development of Geometric Growth as visualized in figure below:

Van Hiele model is based on 5 sequential levels: (Crowley, 2007 and Patsiomitou, & Emvalotis, 2009)



-Visualization: At this level the student can learn geometric shapes holistically and their names without acknowledging their characteristics.

-Analysis: As student reaches this level, he/she can recognize that a geometric shape (learner can already identify) has different parts as angles, sides…

-Informal Deduction: This level is reached as the student can build relations between parts of a shape or compare parts among different shapes.

-Deduction: At this level the student is able to come up with axioms and postulates concerning different geometric shapes in addition to understand a related theory and its converse.

-Rigor: When the student reaches abstract ideas about geometry after he/she fully understood the related concrete theories and ideas, then the non-Euclidean Geometry becomes the field of work.

Van Hiele Model has five properties: (Crowley, 2007)

- Sequential or linear: the student shows progress from one level to a higher level in order; i.e. the student cannot reach the Deduction level without achieving the skills of the previous level – Informal deduction and so on.

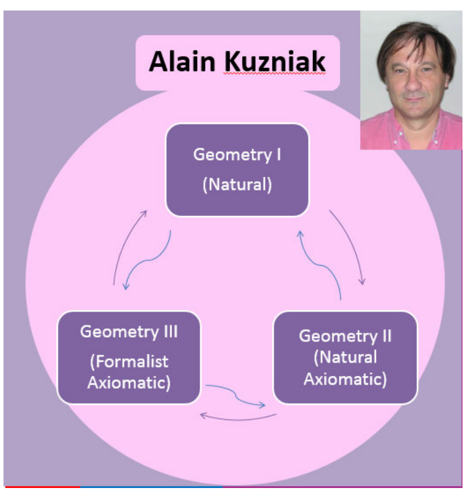
- Advancement: students in this model can learn and acquire skills above their level without knowing; for example they can learn adding fractions without knowing that they are fractions and what do fractions represent.
- Intrinsic and Extrinsic: The idea grasped in level 2, for example, must be the subject for further studies in level 3.
- Linguistics: Each of the levels has its language, vocabulary, symbols and the relations among them,
- Mismatch: If in one level the student is taught a certain skill above his cognitive level and he did not follow the sequence (student capabilities does not match with the student's level) he cannot acquire the knowledge.

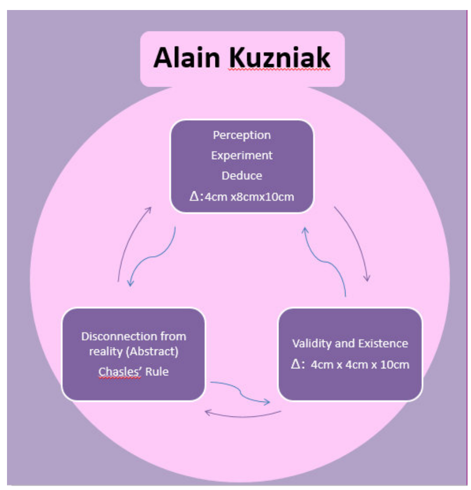
Alain Kuzniak, the French educationist, has a different point of view when it comes to learning Geometry. Kuzniak subdivided Geometry learning into three Geometry zones between which the student will navigate back and forth until the learning is accomplished. (Cerme3, 2003 and Kuzniak et al, 2007)

Kuzniak three levels of Geometry learning can be briefly described as:

- Geometry I (Natural Geometry): In this level of Geometry the reasoning is naturally based on experience and intuition. It is based on concrete facts. At this level Kuzniak gives the priority to the learner's immediate perception; after-which the learner experiments on the material objects and the use of instruments until he/she comes out with a deduction.

Example: To construct a triangle the length of its sides are 4cm, 8cm, and 10cm**, s**tudents can do the task first by playing with sticks, straws or spaghetti bars of the given lengths. Here, the student is also experimenting on the concrete existence of the triangle. Later on, the students can draw the same triangle on a paper using the instruments as ruler and compass.





- Geometry II (Natural Axiomatic): At this level, the student can defend the existence and validity of geometric forms based on the hypothetical deductive laws in an axiomatic system. The necessary system of axioms must be as close as possible to intuition or reality around the learner.

Example: Certain triangles can look strange or even do not exist for a combination of lengths as 4cm, 4cm, and 10cm. Here arises the idea of axiom or relation between lengths of three sides for a triangle to exist.

- Geometry III (Formalist Axiomatic Geometry): At this level the learner has the ability of disconnecting reality from axioms. Learner can extrapolate from the system of axioms to abstract ideas; in other words there is a possibility of working without any relation to concrete reality. In this phase of abstraction, the learner can use similar reasoning to Geometry II but independent from validity or existence or applications in real life. The idea which governs is the absence of contradictions or consistency.

Example: the relation between lengths of sides for a triangle to exist can now be replaced by the vector relation of Chasles' theorem which goes beyond triangles and can be applied to vectors regardless of embedding meanings from real life.

The table below was presented by Kuzniak himself in one of his papers in (Houdement, & Kuzniak, 2003) conference explaining his theory:

	Geometry I (Natural Geometry)	Geometry II (Natural Axiomatic Geometry)	Geometry III (Formalist Axiomatic Geometry)
Intuition	Sensible, linked to the perception, enriched by the experiment	Linked to the figures	Internal to mathematics
Experience	Linked to the measurable space	Linked to schemas of the reality	Logical
Deduction	Near of the Real and linked to experiment	Demonstration based upon axioms	Demonstration based on a complete system of axioms.
Kind of spaces	Intuitive and physical space	Physical and geometrical space	Abstract Euclidean Space
Status of the drawing	Object of study and of validation	Support of reasoning and "figural concept"	Schema of a theoretical object, heuristic tool
Privileged aspect	Self-Evidence and construction	Properties et demonstration	Demonstration and links between the objects. Structure.

The Practical Part:
In what follows two situation problems are considered as examples to defend our hypotheses concerning new reading to improve long term problem solving skills.
(Houdement, C. & Kuzniak, A. (2003) TG7_Houdement_cerme3, p7)

- **Example One:**

Objectives: This problem is designed to be solved with the help of the instructor and without final assessment. The objective of this problem is to verify that it is possible to solve real life problems through Geometry.

The Problem: A helicopter in position A wants to put off the fire in position B after filling its tank (reservoir) from the sea. Help the helicopter pilot to choose the shortest path for his mission in order to save time and energy. (Figure 1.a)

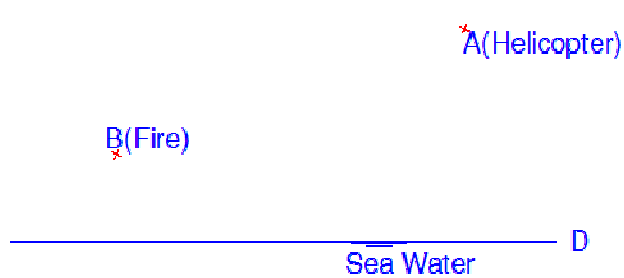
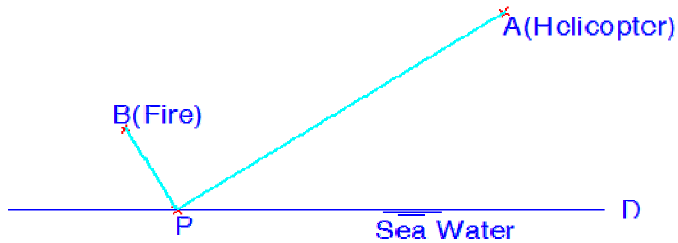


Figure 1.a

Step 1: Let P (any point of your choice) be the possible point on sea surface from which the helicopter can fill its reservoir. Is the path AP + PB the shortest distance between helicopter position (A) and the fire location (B)? (Refer to figure 1.b).

Figure 1.b



Hint 1: Concept of axis of symmetry for B and B'. (Figure 1.c)

Use the concept of axis of symmetry to answer the above question: Is the path AP + PB the shortest distance between helicopter position A and the fire location B?

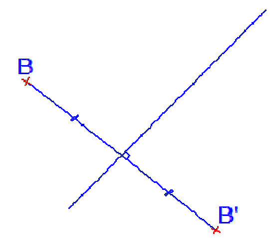


Figure 1.c Axial Symmetry

Step 2: Draw point B' the symmetric of B (fire position) with respect to axis along line D (surface of sea water). Refer to figure 1.d.

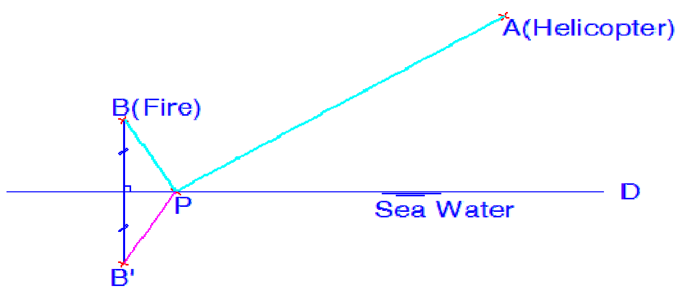
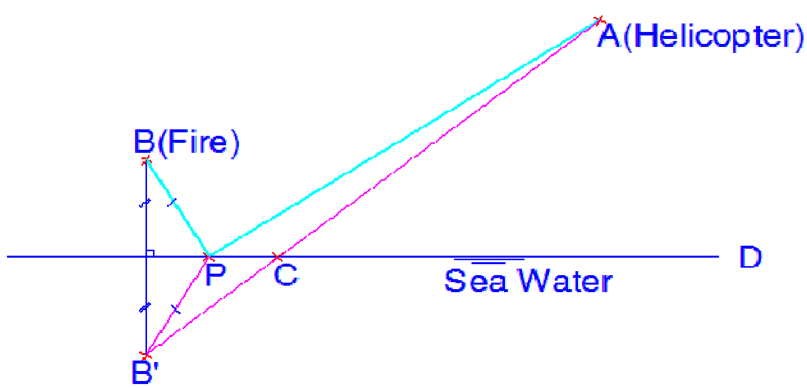


Figure 1.d

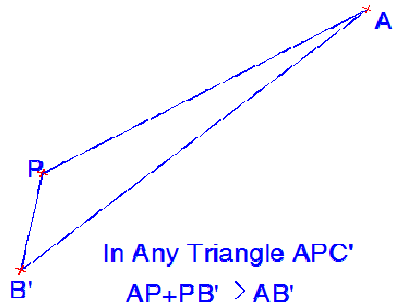
Step 3: Join A to B'. [AB'] intersects line D in point C. Is AB' < (AP + PB) or in other words is (AC + CB') < (AP + PB)? Refer to figure 1.e

Figure 1.e



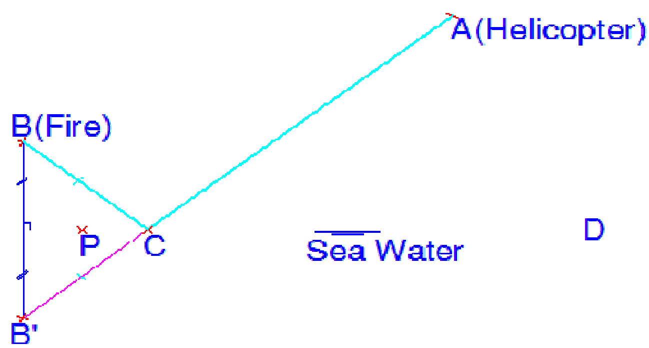
Hint: In any triangle, the sum of the lengths of any two sides is larger than the length of the third side. (Figure 1.f)

Figure 1.f



Then AP + PB' > AB' Or AP + PB' > AC +CB'
But PB' = PB (symmetry)
AP + PB > AB' or AP +PB' > AC + CB'
Therefore, the path of AP+ PB is not the shortest path
.**Step 4:** So, where is the shortest path?
Hint: Is C the critical point we are looking for? (Refer to figure 1.g)

Figure 1.g



Since AC + CB' < AP + PB
And CB = CB" (symmetry)
Then AC + CB < AP + PB

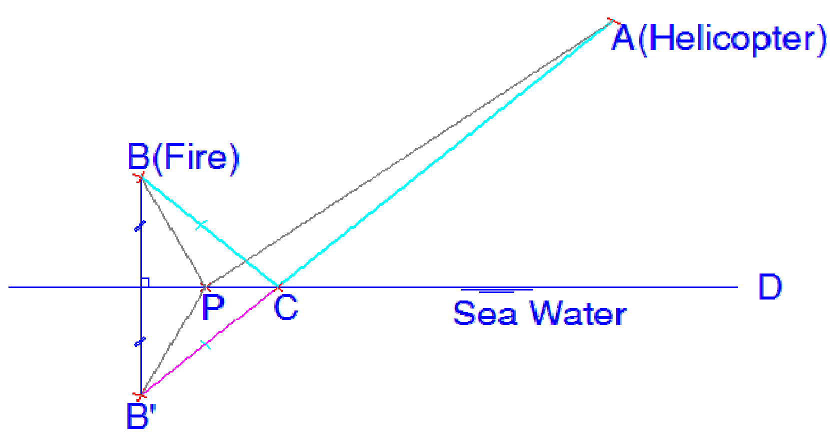


Figure 1.h

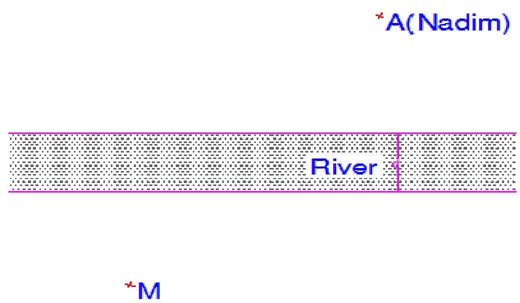
Therefore, AC + CB is the shortest path to be followed by helicopter in order to save time while putting off fire in location B. (Refer to figure 1.h)

- **Example Two:**

Objectives: This problem is designed as a succession to the first problem in order to evaluate the students' acquisition of geometry skills implemented to solve the real life problems and to show the effectiveness of the hints in leading the students through their ZPD and the scaffolding process. (Risen, 2011)

The Problem: Nadim is in position A next to river side; he wants to cross the river of width d to reach the other side of the river to position M. (Refer to figure 2.a). Can you help Nadim to determine where should the location of the bridge be so that he travels the shortest path to reach position M on the other side of the river along path (AP +PQ +QM) where PQ represents the bridge?

Figure 2.a



Note: Students may consider the path joining A to M as the shortest path which is true but only theoretically. (Figure 2.b)

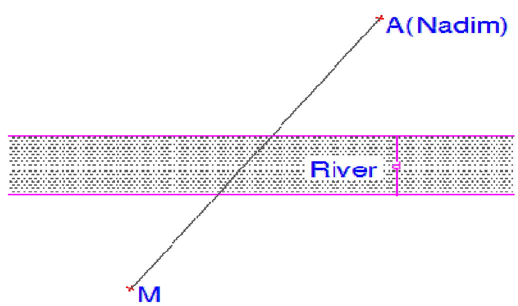
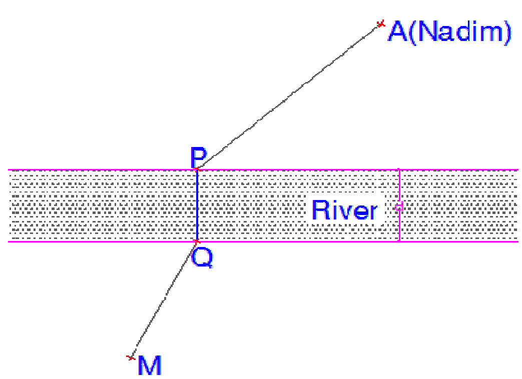


Figure 2.b

Hint: In practice bridges like [PQ] are always constructed over the shortest distance above the water. This means that the bridge is always perpendicular to the two river banks from both sides (assuming the river banks are parallel straight lines as in figure 2.c)

Figure 2.c

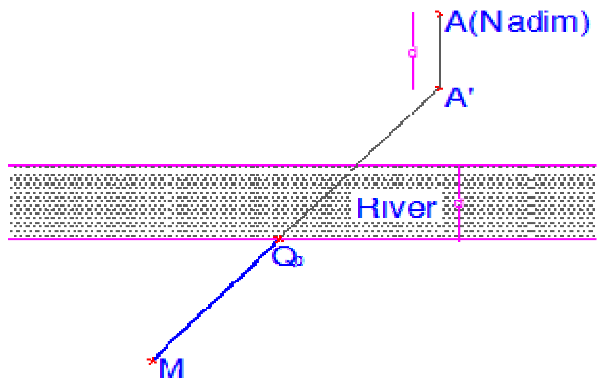


Step 1: Then, where is the best location for the bridge between points A and M so that Nadim can travel the shortest path AP +PQ +QM?

Hint: [AA'] compensates for the width of the river which is [PQ] = d. (Figure 2.d)

Step 2: When AA' compensates for width of river d (AA' = PQ and $\overrightarrow{AA'} = \overrightarrow{PQ}$) and A'=$T_{\overrightarrow{PQ}}$A, then A'M is the least distance between A' and M theoretically. A'M also represents the shortest distance traveled if there were no river.

Figure 2.d



Step 3: AP + PQ + QM = AA' +A'Q + QM since AA'QP is a parallelogram and opposite sides are equal and parallel. (Refer to figure 2.e)

Figure 2.e

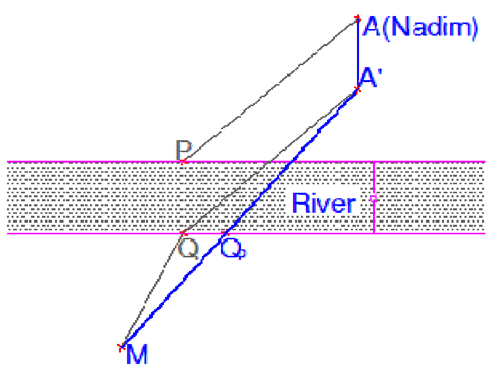
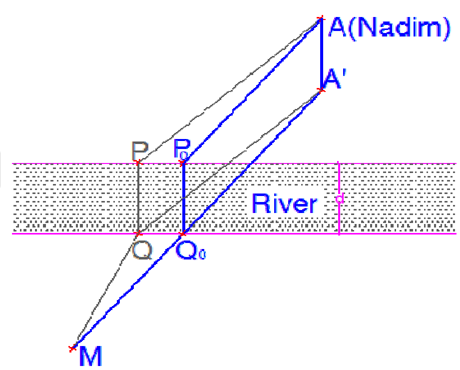


Figure 2.f



A'Q + QM > A'M (In any triangle, the sum of the lengths of any two sides is larger than the length of the third side.)

AP +QM > A'M (A'Q and AP are opposite sides of parallelogram AA'QP)

AP +PQ + QM > A'M + PQ (Add PQ to both sides of inequality)

AP + PQ + QM > A'M + AA' (PQ and AA' are opposite sides of parallelogram AA'QP)

Then AP + PQ + QM is not the shortest path.

The shortest path in distance is A'M + AA'.

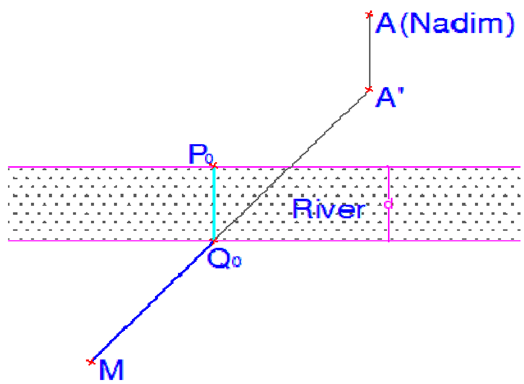
A'M + AA' = (A'Q_0 + Q_0M) + AA' (Q_0 is on line joining A' and M – figure 2.f)

And A'Q_0 + Q_0M + AA' = A'Q_0 + AA' + Q_0M (Addition is commutative)

But A'Q_0 + AA' + Q_0M = AP_0 +$P_0$$Q_0$ +Q_0M (AA'$Q_0$$P_0$ is a parallelogram)

Therefore, The shortest path A'M + AA' = AP_0 +P_0Q_0 +Q_0M
And consequently the bridge should be in position P_0Q_0.
***Note that Q0 should be located before P0
Step 4: How can you construct geometrically the position of Q_0P_0 the optimum location of the bridge?
Construct A' such that AA' compensates for the river width
d.
Draw the line joining A' to M

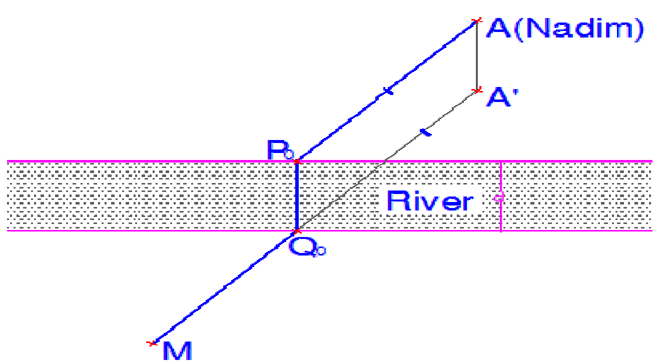
Figure 2.g



The intersection of A'M and the river bank from M –side is point Q_0 the first edge of the bridge. (Figure 2.g) Draw through Q_0 the perpendicular to river banks and determine P_0 as the intersection between the perpendicular drawn and the other river bank (Nadim side).

Step 5: Now [P_0Q_0] represents the bridge and Nadim must trace the path AP_0 then P_0Q_0 then Q_0 M to reach his destination following the shortest path. (Figure 2.h)

Figure 2.h



Data and Analysis:

The graph representing the spiral curriculum as per Bruner indicates several steps to be covered by the student to progress forward. In what follows is an explanation of how the presented approach follows the indicated steps:

- Expectations for students: Grade 9 learners are expected to solve the geometric problems encountered in real life using the theorems and skills mastered at school.

- Content and performance Standards: The curricular books are expected to tackle real life problems to apply the mastered theorems and skills.
- Instructional Practices: Here the major changes are suggested; the teacher is supposed to provide hints and clues considered the SCAFFOLDS at the appropriate times to help the student in his Zone of Proximal Development –ZPD- to fill in a gap and to progress to the higher level question(s).
- Assessment of progress: This step in the spiral curriculum is ongoing when the teacher is always present with the students to ensure their arrival to the safe shore before any further step(s) are taken or admitted.
- Reflections/Evaluations: In this approach a better reflection can be done through metacognition; the teacher can aid the student to store in his long term memory a strategy to be followed in similar cases encountered in the future.
- Modification Based on New Knowledge: The presentation of a similar but more advanced real life problem as a high level experience indicates whether the student has passed his ZPD or s/he still requires more drilling practice. In this paper, Nadim problem to reach destination M comes after the helicopter-sea water-fire mean trajectory.

The example problems chosen in this paper admit the first three levels of Van Hiele's model: In Problem 1, the student uses the visualization in the concept of axis of symmetry and the triangular inequality (inequality relation between the lengths of the three sides of a triangle to exist). In the Analysis level, the student, after given the hint of drawing B'- the symmetric of B with respect to line D ($B' = S_D B$), can connect with this concept the properties of equal sides from a point on axis of symmetry to B and to B' (PB =PB"). In the informal deduction level, the student is building the connections and relations between triangular inequality relation, the equal sides, and the comparison between two paths in order to determine the shortest. In this level, the student is also building the relations between the least time and the shortest path and the save of energy in using the helicopter engine.

In Problem 2, the visualization is demonstrated in acknowledging that the shortest distance between two points is the segment joining them, in the compensation for the river width by AA', and detecting the presence of parallelograms. The Analysis level is embedded in the steps of recognizing the properties of a parallelogram. The informal deduction is demonstrated in all other parts of the solution where the different relations applied one after the other lead to the required comparison of trajectories lengths and consequently the solution.

The solutions to the problems presented in this paper fall under the first two Geometry levels on Kuzniak scale:

In problems 1 and 2, the intuition in Natural Geometry I is by arising student concern with the problem of putting off the fire, guiding the pilot and helping Nadim (Grade 9 virtual student). In the same level, student uses his perception of straight lines are shorter than any other path, and bridges are built along the shortest path over the river.

In the Natural Axiomatic Geometry II, intuition is linked to figures of the concept of symmetry and the condition for the existence of a triangle. Under experience and Geometry I, the given figure of the problem simplifies the wide areas of the fire and sea surface into a point and a line for problem 1 while in problem 2, the figure assumes the river as a band of constant width between parallel lines. In addition, it helps the student in Geometry II to learn how to design schemas for connecting with similar future experiences.

This also brings the student under deduction to solve a geometry problem very close to the real case. The student here does not go into details of whether the helicopter in reality follows the exact geometric path mapped or practically it hovers in its region, or whether Nadim will trace the path walking, cycling, or using a vehicle.

Again the intuitive, physical, and geometric spaces are clear to the student who can identify the difference between them.

The student after reaching a resolution is confident of its validity and is capable of justifying his approach.

Under privileged aspect, the student recognizes that he has to search alone for the evidences that allow him to construct the geometric figure which leads to the solution; in addition the student is finally capable of demonstrating his findings.

Results of Trial:

The presented problems were the subject of study performed on Grade 9 students. The results can be summarized as such: Almost all the students were able to draw the figure of problem 2 but only 30% could solve the problem without the provided hints. After providing the hints, 60% of the students can reach the solution, but there was a problem in identifying the difference between points P_0 and Q_0 representing the edges of the bridge on both sides of the river; students were confused with the backwards strategy of how Q_0 is determined before P_0 although Nadim is supposed to cross the river on the bridge from P_0 to Q_0. Nevertheless, after additional hints, 78% of the students only were able to recognize that P_0 is determined after locating Q_0 which is a much better achievement but still requires further studies attempting to increase this percentage.

Conclusion:

Researchers, teachers, and mathematicians are continuously looking for strategies to help the middle school students, and specifically Grade 9 level, overcome the difficulties of geometry learning. This paper suggests an approach integrating the effectiveness of scaffolding as per Jerome Bruner through the focus on the group-solving supported by hints and continuously assessed by the teacher, and the first three sequential levels of Pierre Van Hiele's model of development of Geometric growth: visualization, analysis, and informal deduction in order to reach a demonstration, in addition to the implementation of Alain Kuzniak levels I and II of Geometry according to his interpretation of geometry learning but with the help of the hints in the presence of the more knowledgeable other- the teacher.

The presented strategy has proved to encourage the students to step into their zones of proximal development region and cross it safely with the scaffolds of their teacher to come out confident with their capacities to solve similar and more advanced problems.

This approach is discussed through two examples flavored by real life cases to attract and motivate the student towards geometric solutions. Data analysis of the trial reflected a progress up to 78% of students able to solve such well-designed examples. The importance of this strategy lies behind considering such examples as a sample problem stored in students' long term memory and retrieved whenever needed to solve similar cases in the future.

This approach suggests adding similar problems to the admitted Math books of the curriculum and asks the Math teachers to solve them in class aiding the students in their mission.

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